Constitutive relations for an isotropic solid
- for a solid, the stress depends on the strain at each instant in time
- recall Hooke's law $F = K \times d$: relationship between force (stress) and extension (strain)
- for small strains, the relationship is usually approximately linear
- the general linear relationship between two tensors (stress $\sigma_{ij}$ and strain $e_{ij}$) is $\sigma_{ij} = C_{ijkl} e_{kl}$ where $C_{ijkl}$ is a fourth rank tensor with $3^4 = 81$ components
- now consider isotropic materials, i.e., materials that show the same behavior independent of orientation
- for an isotropic material, $C_{ijkl}$ must be isotropic. Only two linearly independent fourth rank tensors are possible, $\delta_{ij}\delta_{kl}$ and $\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$.
- so $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ and $\sigma_{ij} = \lambda\delta_{ij} e_{kk} + 2\mu e_{ij}$ defines the constitutive relation
- $\lambda$ and $\mu$ are called the Lamé constants

Alternative forms and notations
- in engineering, $G$ is usually used in place of $\mu$, and is called the shear modulus
- in $(x, y, z)$ coordinates
  
  $\sigma_{xx} = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{xx}$
  
  $\sigma_{yy} = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{yy}$
  
  $\sigma_{zz} = \lambda(e_{xx} + e_{yy} + e_{zz}) + 2Ge_{zz}$
  
  $\sigma_{xy} = 2Ge_{xy}$, $\sigma_{xz} = 2Ge_{xz}$, $\sigma_{yz} = 2Ge_{yz}$
- these equations can be solved for strain in terms of stress, as follows
  
  $e_{ij} = (1/E) [(1 + \nu) \sigma_{ij} - \nu \delta_{ij}\sigma_{kk}]$
  
  where $E$ is the Young's modulus and $\nu$ is the Poisson ratio
  
  $E = G(3\lambda + 2G)/(\lambda + G), \quad \nu = \lambda/[2(\lambda + G)]$
  
  - see Fung for other relationships between these constants
  
  alternative notations for stress: $\sigma_{ij}$, $\tau_{ij}$, $T_{ij}$

Interpretation of Young's modulus and Poisson ratio
- uniaxial stress: suppose that a block of material is stretched in the $z$ direction, with no stresses applied in other directions

\[
\begin{align*}
\sigma_{zz} & \text{ is the only non-zero component of stress} \\
\text{strain is given by } e_{zz} = (1/E) \sigma_{zz}, e_{xx} = (-\nu/E) \sigma_{zz}, e_{yy} = (-\nu/E) \sigma_{zz}\n\end{align*}
\]
• the block has stretched in the z direction and contracted in the other two directions by \( \nu \) times the stretch
• \( E \) is the elastic modulus associated with uniaxial stress, and has dimensions of stress (e.g., Pascal, dyn/cm\(^2\), psi, etc.)
• \( \nu \) is a ratio of contraction, transverse to uniaxial stress, to expansion with the stress, and is a dimensionless ratio

Interpretation of shear modulus
• shear stress: suppose that a block of material is sheared in the x-z plane as shown

\[
e_{xz} = e_{xz} = \frac{1}{2} \frac{du}{dz} \quad \text{is the shear strain and } \quad \sigma_{xz} = \sigma_{xz} = G \frac{du}{dz} \quad \text{is the shear stress, where } G \text{ is the shear modulus, so } \sigma_{xz} = 2G e_{xz}, \text{ etc.}
\]
• in terms of \( E \) and \( \nu \), \( G = E/[2(1 + \nu)] \)

Typical values of elastic parameters
• Young's modulus varies widely for biological tissues
  - bone: 10 - 20 GPa
  - tendon/ligaments: ~0.1 - 1 GPa
  - cartilage: ~10 MPa
  - artery wall: ~1 MPa
  - skeletal muscle: 10 - 20 kPa
  - brain: ~2 kPa
  - tumor tissues: 300 - 3000 Pa
  - white blood cells: ~30 Pa
  - red blood cells: equivalent to ~3 Pa
Note that the Young's modulus of steel is about 200 GPa

• biological tissues have a large water content, water is almost incompressible.
  For incompressibility, \( e_{ii} = (1/E) \left[(1 + \nu) \sigma_{ii} - 3\nu \sigma_{kk}\right] = (1/E) \left(1 - 2\nu\right) \sigma_{ii} = 0 \)
  (write out \( e_{ii} \) to see why factor of 3 appears)
  so Poisson ratio \( \nu = 1/2 \) is often assumed for biological tissues
• but if a tissue is deformed slowly enough, water can move relative to other tissue components, allowing compression of the tissue, and then \( \nu \neq 1/2 \)
• many biological materials are subject to large deformations, and the behavior is nonlinear
• many biological materials are anisotropic, and show very different responses when stretched in different directions